ON $(n+2)$-DIMENSIONAL n-LIE ALGEBRAS

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ABSTRACT. I show that an $(n+2)$-dimensional n-Lie algebra over an algebraically closed field must have a subalgebra of codimension 1.

R. Bai, X. Wang, H. An and W. Xiao [1] have been working on the classification of the 5-dimensional 3-Lie algebras over an algebraically closed field of characteristic 2. To complete their classification, they ask if such an algebra must have a subalgebra of dimension 4. The following theorem answers that question.

Theorem 0.1. *Let A be an $(n+2)$-dimensional n-Lie algebra over the algebraically closed field F. Then A has a subalgebra of codimension 1.*

Proof. We denote the derived algebra $[A, A, \dots, A]$ of A by $A^{(1)}$. If $A^{(1)} < A$, then A has a subalgebra of codimension 1 since any subspace containing $A^{(1)}$ is a subalgebra. Hence we may assume $A^{(1)} = A$ and so, that A is not nilpotent. Let H be a minimal Engel subalgebra. Then $n-1 \le \dim(H) \le n+1$. As F is infinite, H is a Cartan subalgebra by Barnes [2, Theorem 4.3]. If $\dim(H) = n+1$, the result holds, so we may assume $\dim(H) \le n$. This implies that H is abelian and is represented on A by commuting linear transformations. Since F is algebraically closed, they have a common eigenvector u. Thus we have $[h_1, h_2, \dots, h_{n-1}, u] = \alpha(h_1, h_2, \dots, h_{n-1})u$ for all $h_1, h_2, \dots, h_{n-1} \in H$, where α is a linear map $H^{\wedge(n-1)} \to F$. If $\dim(H) = n$, then $\langle H, u\rangle$ is an $(n+1)$-dimensional subalgebra.

Suppose $\dim(H) = n-1$, $H = \langle a_1, \dots, a_{n-1}\rangle$. Let d be the inner derivation $d(a_1, \dots, a_{n-1})$ of A. For each eigenvalue λ of d, we have the λ-component $A_\lambda = \{a \in A \mid (d-\lambda I)^{n+2}a = 0\}$ of A, where I denotes the identity transformation. We have $A_0 = H$ and A is the direct sum of the components for the eigenvalues of d. Let $\lambda_1, \dots, \lambda_n$ be (not necessarily distinct) eigenvalues of d. Then $[A_{\lambda_1}, \dots, A_{\lambda_n}] \subseteq A_{\lambda_1+\dots+\lambda_n}$.

Since $H \subset A^{(1)}$, we either have two eigenvalues, say α, β with sum 0 or we have $\alpha+\beta+\gamma = 0$. Suppose first that $\alpha+\beta = 0$. Suppose $\mathrm{char}(F) \neq 2$. We have eigenvectors u, v for α, β. Then $\langle H, u, v\rangle$ is an $(n+1)$-dimensional subalgebra. Suppose $\mathrm{char}(F) = 2$. Then we can choose u, v such that $[a_1, \dots, a_{n-1}, u] = \alpha u$ and $[a_1, \dots, a_{n-1}, v] = \alpha v + \theta u$ for some θ. Again we have that $\langle H, u, v\rangle$ is an $(n+1)$-dimensional subalgebra.

Now suppose $\alpha+\beta+\gamma = 0$. Suppose $\mathrm{char}(F) \neq 2$. Then $\alpha+\beta$ is not an eigenvalue of d, so $[h_1, \dots, h_{n-2}, u, v] = 0$ for all $h_1, \dots, h_{n-2} \in H$. Thus $\langle H, u, v\rangle$ is an $(n+1)$-dimensional subalgebra of A. Now suppose $\mathrm{char}(F) = 2$. We have

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the distinct non-zero eigenvalues $\alpha, \beta, \gamma = \alpha + \beta$ and corresponding eigenvectors u, v, w. If $n = 2, 3$, then $H \not\subseteq A^{(1)}$, so we may suppose $n \geq 4$.

For some re-ordering of the basis of H, we have $[a_1, \dots, a_{n-3}, u, v, w] \neq 0$. Denote the string $a_1, \dots, a_{n-3}$ by $\overrightarrow{a}$. We apply the Jacobi identity to the product $P = [\overrightarrow{a}, a_{n-2}, u, [\overrightarrow{a}, a_{n-1}, v, w]]$. Since $[\overrightarrow{a}, a_{n-1}, v, w] \in \langle u \rangle$, $P = 0$. But

$$\begin{aligned} P &= [\overrightarrow{a}, [\overrightarrow{a}, a_{n-2}, u, a_{n-1}], v, w] + [\overrightarrow{a}, a_{n-1}, [\overrightarrow{a}, a_{n-2}, u, v], w] \\ &\qquad + [\overrightarrow{a}, a_{n-1}, v, [\overrightarrow{a}, a_{n-2}, u, w]] \\ &= [\overrightarrow{a}, \alpha u, v, w] + 0 + 0 \end{aligned}$$

since $[\overrightarrow{a}, a_{n-2}, u, v] \in \langle w \rangle$ and $[\overrightarrow{a}, a_{n-2}, u, w] \in \langle v \rangle$. Therefore $\alpha = 0$ contrary to the definition of α. Thus this case cannot arise. □

References

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